ALGEBRA.

76. Proposed by E. B. ESCOTT, Fellow in Mathematics, University of Chicago, Chicage, Ill.

Prove the identities

$$2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}\cdots\cdots$$

$$\frac{5-\sqrt{5}}{2}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\cdots\cdots$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let N be any number, then

$$N=\frac{Np^m}{p^m}\cdot\frac{r^m}{r^m}=\frac{p^m}{r^m}\left[1-\left(\frac{p^m-Nr^m}{p^m}\right)\right].$$

$$\therefore \sqrt[m]{N}=\frac{p}{r}\left[1-\frac{p^m-Nr^m}{p^m}\right]^{1/m}=\frac{p}{r}\left[1-\left(\frac{p^m-Nr^m}{mp^m}\right)\right.$$

$$-\frac{(m-1)}{1.2}\left(\frac{p^m-Nr^m}{mp^m}\right)^2-\frac{(m-1)(2m-)}{1.2.3}\left(\frac{p^m-Nr^m}{mp^m}\right)^3$$

$$\left.-\frac{(m-1)(2m-1)(3m--1)}{1.2.3.4}\left(\frac{p^m-Nr^m}{mp^m}\right)^4-\text{ etc.}\right].$$

Let $m=2$ and $p^m-Nr^m=1$,

$$\therefore \sqrt{N}=\frac{p}{r}\left(1-\frac{1}{p^2}\right)^{\frac{1}{2}}=\frac{p}{r}\left[1-\tfrac{1}{2}\left(\frac{1}{p^2}\right)-\frac{1}{2^3}\left(\frac{1}{p^2}\right)^2-\frac{1}{2^4}\left(\frac{1}{p^2}\right)^3\right.$$

$$\left.-\frac{5}{2^7}\left(\frac{1}{p^2}\right)^4-\text{ etc.}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $m=2$ and $p^m-Nr^m=4$,

$$\therefore \sqrt{N}=\frac{p}{r}\left(1-\frac{4}{p^2}\right)^{\frac{1}{2}}=\frac{p}{r}\left[1-\tfrac{1}{2}\left(\frac{4}{p^2}\right)-\frac{1}{1.2.2^2}\left(\frac{4}{p^2}\right)^2\right.$$

$$\left.-\frac{1.3}{1.2.3.2^3}\left(\frac{4}{p^2}\right)^3-\frac{1.3.5}{1.2.3.4.2^4}\left(\frac{4}{p^2}\right)^4-\text{ etc.}\right].$$

$$\therefore \sqrt{N}=\frac{p}{r}\left[1-\frac{2}{p^2}-\frac{2}{p^4}-\frac{4}{p^6}-\frac{2.5}{p^8}-\frac{4.7}{p^{10}}-\frac{3.4.7}{p^{12}}-\text{ etc.}\right]\ldots(2).$$

In (1) let $N=2$, $p=17$, $r=12$, $\therefore p^2-2r^2=1$.

$$\therefore \sqrt{2}=\tfrac{17}{12}\left(1-\frac{1}{2.17^2}-\frac{1}{2^3.17^4}-\frac{1}{2^4.17^6}-\frac{5}{2^7.17^8}-\text{ etc.}\right).$$

$$\therefore\ 2-\sqrt{2}=2-\tfrac{17}{12}+\frac{1}{2.12.17}+\frac{1}{2^3.12.17^3}+\frac{1}{2^4.12.17^5}+\frac{5}{2^7.12.17^7}+\text{ etc.}$$

$$2-\sqrt{2}=\tfrac{1}{2}+\tfrac{1}{12}+\frac{1}{2.12.17}+\frac{1}{2^3.12.17^3}+\frac{1}{2^4.12.17^5}+\frac{5}{2^7.12.17^7}+\text{ etc.}$$

$$=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\left(\frac{1}{2^4.3.17.577}-\frac{1}{2^4.3.17.577.578}\right)$$

$$+\left(\frac{1}{2^4.3.17.577.578}-\frac{1}{2^4.3.17.577.578^2}\right)$$

$$+\left(\frac{1}{2^4.3.17.577.578^2}+\frac{1}{2^5.3.17.577.665857}-\right)+\text{ etc.}$$

$$\therefore\ 2-\sqrt{2}=\tfrac{1}{2}+\frac{1}{2^2.3}+\frac{1}{2^3.3.17}+\frac{1}{2^4.3.17.577}+\frac{1}{2^5.3.17.577.665857}+\cdots\cdots$$

In (2) let $N=5$, $p=7$, $r=3$, $\therefore\ p^2-5r^2=4$.

$$\therefore\ \sqrt{5}=\tfrac{7}{3}\left(1-\frac{2}{7^2}-\frac{2}{7^4}-\frac{4}{7^6}-\frac{2.5}{7^8}-\frac{4.7}{7^{10}}-\text{ etc.}\right)$$

$$\frac{5}{\sqrt{2}}=\tfrac{6}{7}-\frac{1}{3.7}-\frac{1}{3.7^3}-\frac{2}{3.7^5}-\frac{5}{3.7^7}-\frac{2}{3.7^8}-\text{ etc.}$$

$$\frac{5-\sqrt{5}}{2}=\tfrac{5}{2}-\tfrac{7}{6}+\frac{1}{3.7}+\frac{1}{3.7^3}+\frac{2}{3.7^5}+\frac{5}{3.7^7}+\frac{2}{3.7^8}+\text{ etc. }=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}$$

$$+\left(\frac{1}{3.7.47}-\frac{2}{3.7.47.49}\right)+\left(\frac{2}{3.7.47.49}-\frac{4}{3.7.47.49^2}\right)$$

$$+\left(\frac{4}{3.7.47.49^2}+\frac{1}{3.7.47.2207}-\right)+\text{ etc.}$$

$$\therefore\ \frac{5-\sqrt{5}}{2}=\tfrac{1}{1}+\tfrac{1}{3}+\frac{1}{3.7}+\frac{1}{3.7.47}+\frac{1}{3.7.47.2207}\cdots\cdots$$

The above formulæ were used by Dr. Artemas Martin for extracting the root of numbers several years ago.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Since $2-\sqrt{2}$ is rather more than $\frac{1}{2}$, we can, confining ourselves to 4 terms, put $2-\sqrt{2}=\frac{1}{2}+\frac{1}{2^2x}+\frac{1}{2^3xy}+\frac{1}{2^4xyz}$, or by transposition, $\sqrt{2}=\frac{3}{2}-\frac{1}{4x}-\frac{1}{8xy}-\frac{1}{16xyz}$. Squaring and omitting the term $\frac{1}{256x^2y^2z^2}$, we get, after multiplying by 64 x,

$$16x=48-\frac{4}{x}+\frac{24}{y}-\frac{1}{xy^2}+\frac{12}{yz}-\frac{4}{xy}-\frac{2}{xyz}-\frac{1}{xy^2z},$$

whence $x=3$, and substituting this value and multiplying by $3y$, we get

$$4y=68-\frac{1}{y}+\frac{34}{z}-\frac{1}{yz};$$

$\therefore$ $y=17$, and substituting it, and multiplying by $17z$, we get $z=578-1=577$. If another term had been desired we would have annexed to the above series the term $\frac{1}{2^5xyzt}$ and proceeded in the same manner, though for 4 terms, the series is sufficiently convergent to render the value of $2-\sqrt{2}$ correct for at least 8 decimals.

Since $\frac{5-\sqrt{5}}{2}$ is $=1+$ fraction, we put, restricting the series to 4 terms besides 1,

$$\frac{5-\sqrt{5}}{2}=1+\frac{1}{x}+\frac{1}{xy}+\frac{1}{xyz}+\frac{1}{xyzt},$$

whence $\sqrt{5}=3-\frac{2}{x}-\frac{2}{xy}-\frac{2}{xyz}-\frac{2}{xyzt}.$

Squaring and omitting the term $4/x^2y^2z^2t^2$, we have, transposing, suppressing the common factor 4, and multiplying by x,

$$x=3-1/x+3/y-1/xy^2-1/xy^2z^2+3/yz+3/yzt$$
$$-2/xy-2/xyz-2/xyzt-2/xy^2z-2/xy^2z^2t, \text{ whence } x=3.$$

Substituting this value, and multiplying by $3y$, we have

$$y=7-1/y-1/yz^2+7/z+7/zt-2/yz-2/yz^2t; \text{ whence } y=7.$$

Substituting, and multiplying by $7z$, we get

$$z=47-1/z+47/t-2/zt\,; \text{ whence } z=47.$$

Substituting, and multiplying by $47t$, we get $t=2207$.

Such series converge very rapidly. In German works they are called "Theilbruchreihen," signifying *partial fraction series.* Every common fraction may be converted into such a series by the following process.

Let the fraction be $\frac{15}{19}$.

$\therefore$ $\frac{15}{19}=\frac{1}{2}+1/2.2+1/2.2.7+1/2.2.7.10+1/2.2.7.10.19,$

so that, taking successively 1, 2, 3, 4 of these, we get as in continuous fractions, approximate values, as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{11}{14}$, $\frac{221}{280}$. As in continuous fractions, the numerator of the difference of any two consecutive approximate fractions is always $=1$. Supposing the indeterminate equation $15x-19y=1$. Changing $\frac{15}{19}$ by the above method into such a

15)19(2
11)19(2
3)19(7
2)19(10
1)19(19

series, and take the last approximate fraction, viz, $\frac{221}{280}$, $x=280$, $y=221$, will furnish two values; also, the preceding fraction $\frac{11}{14}$, $x=14$, $y=11$.

77. Proposed by G. I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, N. H.

Solve the equation, $(6x^2+x-3)^2-48^2=(x+15)^2$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va., and A. H. BELL, Hillsboro, Ill.

$(6x^2+x-3)^2-(48)^2=(x+15)^2$.

$\therefore 3x^4+x^3-3x^2-3x-210=0$.

$\therefore x_1=2.924412+$,

$x_2=-3.041623-$,

$x_3=2.804395\sqrt{-1}-.158044$.

$x_4=-2.804395\sqrt{-1}-.158044$.

II. Solution by J. MARCUS BOORMAN, Consultative Mechanician, Counsellor at Law, Inventor, Etc., Hewlett, Long Island, New York.

The *real* roots of $x^4+\frac{1}{3}x^3-x^2-x=70$ (the forms to which the given equation reduces) are:

$x=+2.924412\ 149966\ 623189\ 6108(58211$
$x_1=-3.041622\ 694570\ 750484\ 61819(75892$
} true to "(" marks,—*probably* 25 decimals true.

Found thus: x^4 +	$\frac{1}{3}x^3$	$-1x^2$	$-1x$	-70
At sight $x=\pm 3$(near), try	-3.04	$+8.24-$	-22.01	69.9504
	$-2.71-\times 3$	$+7.24-$	-23.01	$0.04-$
Multiply by -3, $-.04$	$+.13$	21.72	69.03	error $+$*
	$.1084+$	$29-$	$.9204$	
	$+8.24-$	-22.01	69.9504	

*The root $\therefore$ is $-2.0316+$ nearly.

Like treatment by $+2.9$ gives 2.925 near true. I get the rest in a more concise way (shorter than Horner's).

The *resulting* equations of above roots are: (the coëfficients appear on *face of computation*)

$0=x^3+(+\text{root}+\frac{1}{3})x^2+8.526990\ 472861\ 28178\text{etc. } x+23.936434,541435,17396+.$

$0=x^3+(-\text{root}+\frac{1}{3})x^2+7.237594\ 384604\ 24939\text{etc. } x-23.914031,334310,10968+.$

$0=(5.966034,\ 84\text{etc})x^2+(1.289496,088257,03238+)x+46.950465,875795,28364+.$

Hence $x^2+0.216122,\ 788722x=-7.869626,\ 49385$ etc.

$\therefore x_1=-0.108\text{ etc.}\pm\sqrt{-7.890+}$.

Shorter and *more accurate* than Horner's method, *especially* for 5th degree and 6th degree equations.

The other two roots are imaginary.